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Almost simple groups with socle Ree(q) acting on finite linear spaces☆

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Abstract

This article is part of a project set up to classify groups and linear spaces where the group acts transitively on the lines of the space. Let G be an automorphism group of a linear space. We know that the study of line-transitive finite linear spaces can be reduced to three cases, distinguishable by means of properties of the action on the point-set: that in which G is of affine type in the sense that it has an elementary abelian transitive normal subgroup; that in which G has an intransitive minimal normal subgroup; and that in which G is almost simple, in the sense that there is a simple transitive normal subgroup T in G whose centraliser is trivial, so that $T \trianglelefteq G \leq Aut(T)$. In this paper we treat almost simple groups in which T is a Ree group and obtain the following theorem:

Let $T \trianglelefteq G \leq Aut(T)$, and let $\mathcal{S}$ be a finite linear space on which G acts as a line-transitive automorphism group. If T is isomorphic to ${}^2G_2(q)$, then T is line-transitive and $\mathcal{S}$ is a Ree unitary space.

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1. Introduction

A *linear space* $\mathcal{S}$ is a set $\mathcal{P}$ of points, together with a set $\mathcal{L}$ of distinguished subsets called lines, such that any two points lie on exactly one line. This paper will be concerned with linear spaces with an automorphism group which is transitive on the lines. This implies that every line has the same number of points and we shall call such a linear space a *regular linear space*. We shall assume that $\mathcal{P}$ is finite and that $|\mathcal{L}| > 1$.

Let G and $\mathcal{S}$ be a group and a linear space, respectively, such that G is a line-transitive automorphism group of $\mathcal{S}$. We further assume that the parameters of $\mathcal{S}$ are given by (b, v, r, k), where b is the number of lines, v is the number of points, r is the number of lines through a point and k is the number of points on a line with $k > 2$. From the assumption that G is transitive on the set $\mathcal{L}$ of lines, it follows that G is also transitive on the set $\mathcal{P}$ of points. This is a consequence of the theorem of Block in [1].

This article is part of a project set up to classify groups and linear spaces where the group acts transitively on the lines of the space. Let G be an automorphism group of a linear space. By [7], we know that the study of line-transitive finite linear spaces can be reduced to three cases, distinguishable by means of properties of the action on the point-set: that in which G is of affine type in the sense that it has an elementary abelian transitive normal subgroup; that in which G has an intransitive minimal normal subgroup; and that in which G is almost simple, in the sense that there is a simple transitive normal subgroup T in G whose centraliser is trivial, so that $T \trianglelefteq G \leq Aut(T)$. Camina, Praeger, Neumann, Spiezia considered the cases where T is isomorphic to one of 26 sporadic simple groups or an alternating group (see [5] and [9]). In this paper, we consider the case where T is isomorphic to $\mathrm{Ree}(q)$, and we prove the following theorem:

Theorem 1.1. *Let G be an almost simple group acting line-transitively on a finite linear space $\mathcal{S}$. If $Soc(G) \cong {}^2G_2(q)$, then $Soc(G)$ is also line-transitive. Thus by [24] $\mathcal{S}$ is a Ree unitary space, where $Soc(G)$ denotes the socle of G, $q = 3^{2n+1}$ and $n \geq 1$.*

In addition, Camina et al. considered the cases in which T is isomorphic to one of the classical linear groups over finite fields with large dimension. We considered the cases in which T is of the Lie rank ≤ 2. Note that the methods of [5] are of very little use for the Lie type simple groups of small rank. But the methods used this paper are effective for them. We would like to make the following conjecture:

Conjecture 1.2. *Let $T \trianglelefteq G \leq Aut(T)$, and let $\mathcal{S}$ be a finite linear space on which G acts as a line-transitive automorphism group. Then T is also line-transitive.*

We know that the above conjecture is borne out when T is of Lie rank ≤ 2 or isomorphic to one of 26 sporadic simple groups or an alternating group. Thus we should study the cases in which $G = T$. There are some results concerning these cases (see [16,18,24]). There are a lot of results about line-transitive linear spaces (refer to [4–10,14–24,26,27]).

The second section describes the notation and definitions and contains several preliminary results concerning the group ${}^2G_2(q)$ and regular linear spaces. In the third section we give the proof of the theorem.

2. Some notation, definitions and preliminary results

Here we gather definitions and notation from finite group theory which are used throughout this paper. Our conventions for expressing the structure of groups run as follows. If X and Y are arbitrary finite groups, then $X \cdot Y$ denotes an extension of X by Y. The expressions $X : Y$ and $X^{\cdot}Y$ denote split and non-split extensions, respectively. The expression $X \times Y$ denotes the direct product of X and Y. The symbol $[m]$ denotes an arbitrary group of order m while Z_m or simply m denotes a cyclic group of that order. Other notation for group structure is standard. In addition, we use the symbol $p^i \| n$ to denote that $p^i | n$ but $p^{i+1} \nmid n$ and the symbol $\mathrm{Fix}_\Omega(K)$ to denote the set of fixed points in Ω of a subgroup K of $\mathrm{Sym}(\Omega)$. Let G be a group. We use the symbol $e(G)$ to denote the number of involutions in G.

Let G is a finite group. A subgroup H is *subnormal* if H is a term in some composition series of G. A group is called *quasisimple* if it is perfect (i.e. its own commutator subgroup) and simple modulo its center. A *component* of G is a quasisimple subnormal subgroup.

Let both G and A be finite groups, and Ω a finite set. A triple (A, G, Ω) is called *exceptional* if it satisfies the following conditions:

(1) G is a normal subgroup of A;
(2) both A and G are transitive permutation groups on Ω;
(3) the diagonal of $\Omega \times \Omega$ is the only common orbit of A and G on $\Omega \times \Omega$.

This definition is equivalent to the following: Let $\alpha \in \Omega$; then every A_α-orbit except $\{\alpha\}$ breaks up into strictly smaller G_α-orbits.

We call the triple (A, G, Ω) *arithmetically exceptional*, if there is a subgroup B of A which contains G, such that (B, G, Ω) is exceptional, and B/G is cyclic. When A is a primitive permutation group of almost simple type, Guralnick, Muller and Saxl have obtained their classification (see [11]). In particular, when $Soc(A) = {}^2G_2(3^a)$, there is the following lemma:

Lemma 2.1 (*Theorem 1.5(d) of [11]*). *Let G be a primitive group of almost simple type, so $L \trianglelefteq A \leq Aut(L)$ with L a simple nonabelian group. Suppose there are subgroups B and G of A with G normal in A and B/G cyclic, such that (B, G, Ω) is exceptional. Let M be a point stabiliser in A. If $L \cong {}^2G_2(3^a)$, then $M \cap T = {}^2G_2(3^{a/b})$ with $a > 1$ odd and b a prime divisor of a other than* 3 *and* 7.

Now we begin with stating some fundamental properties of ${}^2G_2(q)$, where $q = 3^{2n+1}$ and $n \geq 1$.

The Ree groups ${}^2G_2(q)$ form an infinite family of simple groups of Lie type, and were defined in [25] as subgroups of the group $GL(7, q)$. The order of ${}^2G_2(q)$ is $q^3(q^3 + 1)(q - 1)$. Set $t = 3^{n+1}$ so that $t^2 = 3q$. We give the following information concerning subgroups of ${}^2G_2(q)$. For each l dividing $2n+1$, ${}^2G_2(3^l)$ denotes the subgroup of ${}^2G_2(q)$ consisting of all matrices in ${}^2G_2(q)$ with entries in the subfield of order 3^l. We use the symbols Q and K to denote a Sylow 3-subgroup and a cyclic subgroup of order $q - 1$ of ${}^2G_2(q)$, respectively. Let A_0, A_1, A_2, A_3 denote the cyclic subgroups of order

Table 1
The maximal subgroups of ${}^2G_2(q)$

Group	Structure	Remarks
P_1	$Q : K$	The normaliser of Q in ${}^2G_2(q)$
P_2	$(Z_2^2 \times D_{(q+1)/2}) : Z_3$	The normaliser of a fours-group
P_3	$Z_2 \times PSL(2, q)$	An involution centraliser
P_4	$Z_{q-t+1} : Z_6$	The normaliser of Z_{q-t+1}
P_5	$Z_{q+t+1} : Z_6$	The normaliser of Z_{q+t+1}

$(q-1)/2$, $(q+1)/4$, $q+t+1$, $q-t+1$, respectively. Clearly, any two of the integers 2, 3, $(q-1)/2$, $(q+1)/4$, $(q+t+1)$, $(q-t+1)$ are relatively prime.

Lemma 2.2 (*Theorem C of [12]*). *Let $T = {}^2G_2(q) \trianglelefteq G \leq Aut(T)$ and M be maximal in G and contain no T. Then either $M \cap T$ is conjugate to $P_6(l) = {}^2G_2(3^l)$ for some prime divisor l of $2n+1$, or $M \cap T$ is conjugate to one of the subgroups P_i in Table 1.*

Conversely, if H is conjugate to one of these groups, then $N_G(H)$ is maximal in G.

As regards the subgroups of the Ree group ${}^2G_2(q)$, we have the following results.

Lemma 2.3 (*Levchuk and Nuzhin [13]*). *A solvable subgroup $S \leq {}^2G_2(q) = G$ is conjugate to a subgroup of one of the following groups: $N_G(A_i), i = 0, 1, 2, 3$, $N_G(S_p)$, $p = 2, 3$, where A_i is described as before and S_p is a Sylow p-subgroup of the group ${}^2G_2(q)$.*

Lemma 2.4 (*Levchuk and Nuzhin [13]*). *A non-solvable subgroup $S \leq {}^2G_2(q)$ is isomorphic to one of the following groups: $PSL(2, 8)$, $PSL(2, q')$, $2 \times PSL(2, q')$ $(q' > 3)$, ${}^2G_2(q')$, where q is a power of q'.*

Clearly,

$$\begin{aligned} |{}^2G_2(q)| &= q^3(q^3+1)(q-1) \\ &= q^3(q+1)(q-1)(q+1+t)(q+1-t) \\ &= 2^3 3^{3(2n+1)} \left(\frac{q-1}{2}\right)\left(\frac{q+1}{4}\right)(q+1+t)(q+1-t) \\ &= 2^3 3^{3(2n+1)} |A_0||A_1||A_2||A_3|. \end{aligned}$$

Lemma 2.5 (*Zhou et al. [27]*). *Let P be a Sylow p-subgroup of $G = {}^2G_2(q)$. We have:*

(1) *If $p = 2$, then $N_G(P) \cong Z_2^3 : Z_7 : Z_3$.*
(2) *If $p = 3$, then $N_G(P) \cong [q^3] : Z_{q-1}$.*
(3) *If $p \mid (q-1)/2$, then $N_G(P) = D_{2(q-1)}$.*
(4) *If $p \mid (q+1)/4$, then $N_G(P) = (Z_2^2 \times D_{(q+1)/2}) : Z_3$.*

(5) *If* $p \mid (q+t+1)$, *then* $N_G(P) = Z_{q+t+1} : Z_6$.
(6) *If* $p \mid (q-t+1)$, *then* $N_G(P) = Z_{q-t+1} : Z_6$.

Lemma 2.6. *Let c be an odd prime. Then:*

(i) *if* q_0 *is a proper power of* 3 *and* $\epsilon = \pm$, *then* $(q_0^2 - q_0 + 1, q_0^c + \epsilon 1) = 1$ *unless* $c = 3$ *and* $\epsilon = +$, *and*
(ii) *the equality*

$$3^{3x} + 1 = 4c^y$$

where x *is an odd positive integer and* y *a positive integer has only solution the* $x = y = 1, c = 7$.

Proof. (i) Suppose that there is an odd prime s, such that $s \mid (q_0^2 - q_0 + 1, q_0^c + \epsilon 1)$. If $\epsilon = +$, then either $s \mid (q_0 + 1)$ or $s \mid (q_0^{c-1} - q_0^{c-2} + \cdots - q_0 + 1)$. If the former occurs, then $q_0^2 - q_0 + 1 \equiv 3 \pmod{s}$, from which we deduce $s = 3$. This conflicts with q_0 being a proper power of 3. If the latter occurs, then $c = 3$. In fact, if $c \equiv 1$ or $2 \pmod 3$, then $s = 1$ or $s \mid (q_0 - 1)$. Since $s \mid (q_0^2 - q_0 + 1)$, we have $s = 1$, a contradiction. Thus $\epsilon = -$. Since $q_0^c + 1 = (q_0 + 1)(q_0^{c-1} - q_0^{c-2} + \cdots - q_0 + 1)$, we have

$$q_0^c + 1 \equiv \begin{cases} q_0 + 1 & \pmod{q_0^2 - q_0 + 1} \text{ if } 3 \mid (c-1) \\ 2 - q_0 & \pmod{q_0^2 - q_0 + 1} \text{ if } 3 \mid (c-2). \end{cases}$$

Thus s divides $(q_0 + 1)(2 - q_0)$, contradicting $s \mid (q_0^2 - q_0 + 1)$.

(ii) Since $3^{3x} + 1 = (3^x + 1)(3^{2x} - 3^x + 1)$ and $4 \| (3^x + 1)$, we have $3^x + 1 = 4c^y$ and $3^{2x} - 3^x + 1 = 1$, or $3^x + 1 = 4$ and $3^{2x} - 3^x + 1 = c^y$. This forces $x = y = 1$ and $c = 7$. $\square$

Lemma 2.7. *Let* $q = q_0^c$, *where* q *is an odd power of* 3 *and* $c > 3$ *is an odd prime. Then:*

(i) *each of the integers* $q - 1, q + 1, q - t + 1$ *and* $q + t + 1$ *does not divide* $|{}^2G_2(q_0)|$, *where* t *is as defined before, and*
(ii) *there exists a prime* p *such that* p *divides* $q_0^3 + 1$ *but not* $(q^3 + 1)/(q_0^3 + 1)$ *unless* $c = 7$. *Furthermore, if* $c \neq 3$, *we may assume that* p *divides* $q^2 - q + 1$.

Proof. (i) Since any two of the integers 2, 3, $(q-1)/2$, $(q+1)/4$, $(q+t+1)$, $(q-t+1)$ are relatively prime, we have $(q_0(q_0^2 - 1), q + \epsilon t + 1) = 1$. If $q + \epsilon t + 1$ divides $|{}^2G_2(q_0)|$, then $q + \epsilon t + 1$ divides $q_0^2 - q_0 + 1$. But when $c > 3$, we have $q + \epsilon t + 1 \geq q_0^2 - q_0 + 1$ which is a contradiction. Similarly, we may prove that both $q + 1$ and $q - 1$ do not divide $|{}^2G_2(q_0)|$.

(ii) Suppose that s is a common divisor of $q_0^3 + 1$ and $(q^3 + 1)/(q_0^3 + 1)$. Then s is also a common divisor of $q_0^3 + 1$ and $q_0^{3(c-1)} - q_0^{3(c-2)} + \cdots - q_0^3 + 1$. This forces $s = c$. In this case, if $q_0^3 + 1 = 4c^y$ for some positive integer y, then, by Lemma 2.6, we have $y = 1$ and $c = 7$. Thus when $c \neq 7$, it is certain that there is a prime p other than 2 dividing $q_0^3 + 1$ and not $(q^3 + 1)/(q_0^3 + 1)$. Furthermore, if p does not divide $q^2 - q + 1$, then $q_0^3 + 1$ divides $q + 1$, and so $c = 3$ by Lemma 2.6(i). $\square$

From now on we suppose that G is a line-transitive automorphism group of a linear space $\mathcal{S}$. Thus $\mathcal{S}$ is a regular linear space. We assume that the parameters of $\mathcal{S}$ are given by (b, v, r, k) where b is the number of lines, v is the number of points, r is the number of lines through a point and k is the number of points on a line. Recall the basic counting lemmas for linear spaces:

$$v = r(k-1) + 1 \tag{1}$$

$$v(v-1) = bk(k-1). \tag{2}$$

Let

$$b_1 = (b, v), \quad b_2 = (b, v-1), \quad k_1 = (k, v), \quad \text{and} \quad k_2 = (k, v-1).$$

By (2), $k \mid v(v-1)$ and $b \mid v(v-1)$. Thus $k = k_1k_2$ and $b = b_1b_2$. By (1), we have $(r, b_1k_1) = 1$, together with $(v, b_2k_2) = 1$ and $vr = bk = b_1k_1 \cdot b_2k_2$, and so we get $r = b_2k_2$ and $v = b_1k_1$.

In [5], the authors defined a *significant prime* which divides b but not v. In fact, every prime divisor of b_2 is a significant prime. Thus every linear space other than the projective plane has significant primes.

Let L be a line of $\mathcal{S}$. Then G_L will denote the setwise stabiliser of L in G.

There are two facts that we are going to use throughout this article. The first is that any involution of G is conjugate to some involution of G_L, since any involution fixes at least one line. The other observation is that if an involution in G does not fix a point then the G acts flag-transitively; see [8]. But the flag-transitive linear spaces are classified by Buekenhout, Delandtsheer, Doyen, and so on (see [2] and [3]), and so we assume that every involution fixes at least a point.

The following lemmas are very useful when studying the linear spaces with line-transitive automorphism groups.

Lemma 2.8 (*Zhou et al. [27]*). *Let G act line-transitively on a linear space $\mathcal{S}$. Let K be a subgroup of G. If $K \not\leq G_L$ for any line $L \in \mathcal{L}$, and $K \leq G_\alpha$ for some point $\alpha \in \mathcal{P}$, then $N_G(K) \leq G_\alpha$.*

Lemma 2.9 (*Lemma 2.8 of [22]*). *Let G act line-transitively on a linear space $\mathcal{S}$. If there exists a prime number p such that $p|b$ but $p \nmid v$, then for some $\alpha \in \mathcal{P}$, $N_G(P) \leq G_\alpha$, where P is a Sylow p-subgroup of G.*

Lemma 2.10 (*Liu [19] and [20]*). *Let G act line-transitively on a linear space $\mathcal{S}$. Assume that P is a Sylow p-subgroup of G_α for some $\alpha \in \mathcal{P}$. If P is not a Sylow p-subgroup of G, then there exists a line L through α such that $P \leq G_L$.*

The following result is useful in calculating the number of fixed points of an element.

Lemma 2.11. *Let G be a transitive group on Ω, and K be a conjugacy class of an element of G. Let $x \in K$ and $\mathrm{Fix}_\Omega(\langle x\rangle)$ denote the set of fixed points of $\langle x\rangle$ acting on Ω. Then*

$$|\mathrm{Fix}_\Omega(\langle x\rangle)| = |G_\alpha \cap K| \cdot |\Omega|/|K|,$$

where $\alpha \in \Omega$. In particular, if G has a unique conjugacy class of involutions, then

$$|\mathrm{Fix}_{\Omega}(\langle i \rangle)| = \frac{e(G_{\alpha}) \cdot |\Omega|}{e(G)},$$

where i is an involution of G and $e(G)$ denotes the number of involutions of G.

Proof. Count the number of ordered pairs (α, x), where $\alpha \in \mathrm{Fix}_{\Omega}(\langle x \rangle)$. □

Consider the cycle decomposition of an involution acting on $\mathcal{P}$; we have discovered the following lemma. This is a very useful inequality and plays an important role in the proof of the theorem.

Lemma 2.12. *Let G act line-transitively on a linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ with parameters (b, v, r, k). Let i be an involution of G_L, where L is a line of $\mathcal{S}$. Set $f_1 = |\mathrm{Fix}_{\mathcal{P}}(\langle i \rangle)|$ and $f_2 = |\mathrm{Fix}_{\mathcal{L}}(\langle i \rangle)|$. Then the following cases hold:*

(i) *if $f_1 = 0$ or $\mathrm{Fix}_{\mathcal{P}}(\langle i \rangle) \subseteq L$ for some line L and k even, then $f_2 = v/k$;*
(ii) *if $f_1 = 1$, then $f_2 = (v-1)/k$ or $(v-1)/(k-1)$ according to whether k is even or not;*
(iii) *otherwise, $f_2 > v/k$.*

However, in all cases, the inequality $f_2 > v/k - 1$ always holds.

Proof. Consider the cycle decomposition of i acting on $\mathcal{P}$. We know that i has $(v - f_1)/2$ cycles of length 2. The involution i fixes f_2 lines of $\mathcal{S}$, say L_j, where $1 \le j \le f_2$. Set

$$i = (\alpha_1, \beta_1) \ldots (\alpha_s, \beta_s),$$

where $s = (v - f_1)/2$. Obviously, the assertions (i) and (ii) are true. Therefore, we assume that $f_2 \ge 2$. For any $\alpha \in \mathcal{P}$, if $\alpha \notin \mathrm{Fix}_{\mathcal{P}}(\langle i \rangle)$, then there is $t, 1 \le t \le s$, such that $\alpha \in \{\alpha_t, \beta_t\}$. Thus $\alpha \in A := L_1 \cup \cdots \cup L_{f_2}$. If $\alpha \in \mathrm{Fix}_{\mathcal{P}}(\langle i \rangle)$, then there exists another point $\beta \in \mathrm{Fix}_{\mathcal{P}}(\langle i \rangle)$ such that the line through both α and β is fixed by i. Thus we have $\alpha \in A$. In this case, however, we have $\mathcal{P} \subseteq A$, and so $\mathcal{P} = A$. Note that $|A| \le f_2 \cdot |L_1| = f_2 k$. Thus $f_2 \ge v/k$. Clearly, the equality holds if and only if the intersection of any two lines of L_j $(1 \le j \le f_2)$ is empty, that is, $\mathrm{Fix}_{\mathcal{P}}(\langle i \rangle) \subseteq L$ for some line L and k even. □

Lemma 2.13. *Let G act line-transitively on a linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ with parameters (b, v, r, k). Let i be an involution of G_L, where L is a line of $\mathcal{S}$. Set $f_1 = |\mathrm{Fix}_{\mathcal{P}}(\langle i \rangle)|$ and $f_2 = |\mathrm{Fix}_{\mathcal{L}}(\langle i \rangle)|$. If $\mathcal{S}$ is not a projective plane and $f_1 \ge 2$, then $v \le f_2^2$.*

Proof. Since $\mathcal{S}$ is not a projective plane, we have $r \neq k$. Thus $(v-1)/(k-1) = r \ge k+1$, and so $v \ge k^2$. Since $f_2 \ge 2$, by Lemma 2.12, we have $f_2 \ge v/k$. Hence $v \ge (v/f_2)^2$, that is $v \le f_2^2$. □

Finally, we introduce a very interesting result obtained by Dr. Gill (see [10]).

Lemma 2.14. *Let G act point-transitively on a finite projective plane $\mathcal{S}$. Then either $\mathcal{S}$ is a Desarguesian plane of order q, and $PSL(3, q) \le G \le P\Gamma L(3, q)$ or G has no component.*

3. The proof of the theorem

First, we shall prove the following three lemmas.

Lemma 3.1. *Let G act transitively on a finite set Ω. Suppose that there exists a maximal subgroup H of G such that H is normal in G. If H is not transitive, then $H_\alpha = G_\alpha$ for some $\alpha \in \Omega$.*

Proof. Suppose that there exists an element $g \in G_\alpha$ but $g \notin H_\alpha$. Then $G = HG_\alpha$ (note that here H is maximal subgroup of G). Thus

$$G/H = HG_\alpha/H \cong G_\alpha/(H \cap G_\alpha) = G_\alpha/H_\alpha,$$

which implies that

$$|G : G_\alpha| = |H : H_\alpha|.$$

Therefore H acts transitively on Ω, a contradiction. $\square$

The following lemma was obtained by Dr. Gill. The authors thank him for this result. Because its proof is not very difficult, we give it again.

Lemma 3.2 ([10]). *Let G act line-transitively on a finite linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$. If H is a normal subgroup of G and $|G : H| = s$, a prime, but H is not line-transitive, then either $(G, H, \mathcal{P})$ is exceptional or $\mathcal{S}$ is a projective plane.*

Proof. Since H is not line-transitive and H is maximal in G, we have $G_L = H_L$ by Lemma 3.1. We have two possibilities:

Suppose that H is point-transitive on $\mathcal{S}$. Then let α and β be members of $\mathcal{P}$. Let L be the line through them. Then, since $G_{\alpha,\beta} \leq G_L$ and $H_{\alpha,\beta} \leq H_L$, we know that $G_{\alpha,\beta} = H_{\alpha,\beta}$. Note that $|G_\alpha : H_\alpha| = s$; thus we may conclude that, for all pairs of points α and β, $|H_\alpha : H_{\alpha,\beta}| < |G_\alpha : G_{\alpha,\beta}|$. In other words, considering H and G as permutation groups on $\mathcal{P}$, the only common orbital of H and G is the diagonal. In this situation we get that the triple $(G, H, \mathcal{P})$ is exceptional.

Suppose that H is not point-transitive on $\mathcal{S}$. Then, by the Frattini argument, $G = N_G(P)H$ for all $P \in Syl_p(H)$, where p is any prime dividing $|H|$. If $N_G(P) \leq G_\alpha$, then $G = G_\alpha H$, and so H is point-transitive, which is a contradiction. Thus, by Lemma 2.9, $P \subseteq G_\alpha$, $P \in Syl_p(H)$ implies that $P \subseteq G_L$.

New let $b_H = |H : H_L|$ and $v_H = |H : H_\alpha|$. If p divides $|H|$ but not v_H, then H_α contains a p-Sylow subgroup P of H, and so $P \leq H_L$. This implies that $p \nmid b_H$. In other words, the primes dividing b_H must be the divisors of v_H. Furthermore $b = sb_H$ and $v = sv_H$ (note that here $G_L = H_L$ and $G_\alpha = H_\alpha$ by Lemma 3.1). Thus the primes dividing b must be the divisors of v. Thus there are no significant primes and $\mathcal{S}$ is a projective plane. $\square$

The following result is interesting. It is useful for the proof of Theorem 1.1.

Lemma 3.3. *Let $G = T : \langle x \rangle$ and act line-transitively on a finite linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L})$. Then T acts transitively on $\mathcal{P}$.*

Proof. By [1], we know that G is point-transitive. If $\mathcal{S}$ is a projective plane, then by Lemma 2.14, we have $PSL(3, q) \leq T$, whence T is transitive on $\mathcal{P}$. If T is line-transitive on $\mathcal{S}$, then by [1], T is point-transitive. Therefore, we may assume that $\mathcal{S}$ is not a projective plane and T is not line-transitive.

Let $o(x) = n > 1$ and s be a prime divisor of n. Then there exists a normal subgroup H of G such that $|G/H| = s$ and H is not line-transitive (otherwise replace G by H). Let $y = x^{n/s}$. Then $G/H = \langle yH \rangle$. By Lemma 3.2, the triple $(G, H, \mathcal{P})$ is exceptional. Thus by Lemma 3.3 of [11], we know that y fixes exactly one point of $\mathcal{P}$, say α. Consider the cycle decomposition of x acting on $\mathcal{P}$, and we find that x fixes no other points of $\mathcal{P}$ except α. So $x \in G_\alpha$ and $G = TG_\alpha$, which implies that T acts transitively on $\mathcal{P}$. □

Now we may prove our main theorem.

Since $T = {}^2G_2(q) \trianglelefteq G \leq Aut({}^2G_2(q))$, we may assume that $G = T : \langle x \rangle$, where $x \in Out(T)$, the outer automorphism group of T which may be generated by an automorphism of the field. Let $q = 3^a$, $a > 1$ odd, and $o(x) = m$. Then $m|a$ and $|G| = q^3(q^3 + 1)(q - 1)m$.

Suppose that T is not line-transitive on $\mathcal{S}$. Then there exists a group H such that $T \trianglelefteq H \triangleleft G$ where $|G : H|$ is a prime, H is not line-transitive (otherwise replace G by H). By Lemma 3.2, $(G, H, \mathcal{P})$ is an exceptional triple or $\mathcal{S}$ is a projective plane. By Lemma 2.14 or [23], the latter cannot occur. Note that T is not line-transitive but point-transitive (by Lemma 3.3); hence we have $x \notin G_L$ but $x \in G_\alpha$, and it follows by Lemma 2.8 that we get $N_G(\langle x \rangle) \leq G_\alpha$. Because $C_T(x) \leq N_G(\langle x \rangle)$ and ${}^2G_2(3) \leq C_T(x)$, by Lemmas 2.3 and 2.4, we have $C_T(x) \cong {}^2G_2(q_0)$ for some odd integer c with $q = q_0^c$. It follows that ${}^2G_2(q_0) \leq G_\alpha \cap T = T_\alpha$. Since the only overgroups of ${}^2G_2(q_0)$ are subfield groups, we may assume that $T_\alpha = {}^2G_2(q_0)$, and so $G_\alpha = T_\alpha : \langle x \rangle$. Note that if c is an odd prime, then G_α is a maximal subgroup of G and so G is primitive on $\mathcal{P}$. Thus, by Lemma 2.1, we know that $c \neq 3$ and 7. Clearly, G_α contains no Sylow 3-subgroups of G. Thus, by Lemma 2.10, $Q_0 \leq G_L$, and so $Q_0 \leq T \cap G_L = T_L$, where Q_0 is a Sylow 3-subgroup of ${}^2G_2(q_0)$.

Thus, by Lemmas 2.3 and 2.4, we know that the group T_L is conjugate to $PSL(2, q_1)$, $Z_2 \times PSL(2, q_1)$, ${}^2G_2(q_2)$ or a subgroup of QK, where q_0^3 divides q_1, q_0 divides q_2 and Q and K are described as before.

Let i be an involution of G_L. Note that here all involutions of G_L and G are contained in T_L and T, respectively. By Lemma 2.11, we get

$$f_1 = |\mathrm{Fix}_{\mathcal{P}}(\langle i \rangle)| = \frac{v \cdot e(G_\alpha)}{e(G)} = \frac{v \cdot e(T_L)}{e(T)} = \frac{q(q^2 - 1)}{q_0(q_0^2 - 1)} > 2.$$

Assume that $T_L = {}^2G_2(q_2)$. Then, by Lemma 2.11, we get

$$f_2 = |\mathrm{Fix}_{\mathcal{L}}(\langle i \rangle)| = \frac{b \cdot e(G_L)}{e(G)} = \frac{b \cdot e(T_L)}{e(T)} = \frac{q(q^2 - 1)m}{q_2(q_2^2 - 1)m_1} \leq \frac{q(q^2 - 1)m}{q_0(q_0^2 - 1)m_1} \quad (3)$$

where $|G_L| = |T_L| \cdot m_1$ and so $m_1|m$. By Lemma 2.13, we get the following inequality:

$$v = \frac{q^3(q^3 + 1)(q - 1)}{q_0^3(q_0^3 + 1)(q_0 - 1)} < \left(\frac{q(q^2 - 1)m}{q_0(q_0^2 - 1)m_1} \right)^2,$$

that is,

$$\frac{q(q^2-q+1)}{q_0(q_0^2-q_0+1)} < \frac{(q^2-1)m^2}{(q_0^2-1)m_1^2} < \frac{q^2a^2}{q_0^2-1}.$$

It follows that

$$q^2-q+1 < q_0qa^2.$$

That is,

$$3^{2a}-3^a+1 < 3^{(c+1)a/c}\cdot a^2. \tag{4}$$

If a is a prime, then $c=a$. Thus, by (4), we have

$$3^{2a}-3^a+1 < 3^{a+1}\cdot a^2,$$

which forces $a=3$, a contradiction since $c\neq 3$. Hence a is not prime. In this situation we have $c\geq 5$ and $a\geq 15$ (since a is odd and $c|a$), and so $a<3^{a/5}$. It follows by (4) that

$$3^{2a}-3^a+1 < 3^{8a/5},$$

which is impossible since $a\geq 15$. This contradiction shows that G_L contains no ${}^2G_2(q_2)$.

Assume that $T_L = Q_1K_1$ or U, where $Q_1\leq Q$, $K_1\leq K$, and $U=PSL(2,q_1)$ or $Z_2\times PSL(2,q_1)$. Then

$$b=\frac{q^3(q^3+1)(q-1)m}{|Q_1||K_1|m_1} \quad \text{or} \quad \frac{q^3(q^3+1)(q-1)m}{|U|m_1}, \tag{5}$$

where $m_1|m$.

Suppose that c is a prime. Let M_j be a cyclic subgroup of order $q_0+(-1)^jt_1+1$, where $j=1$ or 2, and $t_1^2=3q_0$. If $M_j\leq G_L$ for $j=1,2$, then $M_j\leq T_L$. Thus $M_j\leq K$ or U. It follows that $|M_j|$ divides $q(q^2-1)$. Thus $|M_j|=1$ by Lemma 2.6(i). But $|M_j|=1$ if and only if $q_0=3$ and $j=1$, and so there is a subgroup M_j, such that $1\neq M_j\leq G_\alpha$ but $M_j\not\leq G_L$. By Lemmas 2.5 and 2.8, we get $Z_{q+(-1)^jt+1}:Z_6\leq N_G(M_j)\leq T_\alpha$, which conflicts with Lemma 2.7(i). This contradiction shows that T is line-transitive.

Now suppose that c is not prime. By Lemma 2.11, we have

$$f_2=|\mathrm{Fix}_{\mathcal{L}}(\langle i\rangle)|=\frac{b\cdot e(G_L)}{e(G)}=\frac{|G|\cdot e(T_L)}{|G_L|\cdot e(T)}<\frac{|G|}{e(T)}=q(q^2-1)m.$$

Since $v\leq f_2^2$, we have

$$v=\frac{q^3(q^3+1)(q-1)}{q_0^3(q_0^3+1)(q_0-1)}<(q(q^2-1)m)^2,$$

from which we deduce that

$$q(q^2-q+1) < (q^2-1)m^2\cdot q_0^7.$$

Thus

$$q^2 < (q+1)m^2q_0^7,$$

and so

$$q - 1 < m^2 q_0^7.$$

It follows that

$$3^{a(c-7)/c} \leq a^2. \tag{6}$$

When $a \geq 35$, $a^2 < 3^{a/5}$. Thus

$$\frac{c-7}{c} < \frac{1}{5},$$

which is impossible since c is odd and not prime. Thus $a < 35$. Since c is not prime, we know that a cannot be prime. This implies that $a \in \{9, 15, 21, 25, 27, 33\}$ and $c = a$ unless $a \neq 27$. When $a = 27$, $c = 9$ or 27. However, these values of a and c do not satisfy (6) except $a = 9$.

In order to complete the proof of Theorem 1.1, it suffices to exclude the case where $a = 9 = c$. We have claimed that $T_L \neq {}^2G_2(q_2)$ above. If $T_L \leq QK$, then we may get $7|b$ but not v. Thus by Lemma 2.9 $N_T(P) \leq G_\alpha$, where P is a Sylow 7-subgroup of G_α. Since $(7, a) = 1$, we have the Sylow 7-subgroups of T are also that of G, and so by Lemma 2.5, $(Z_2 \times D_{(q+1)/2}) : Z_3 \leq G_\alpha$, which is impossible by Lemma 2.7. If $T_L = PSL(2, q_1)$ or $Z_2 \times PSL(2, q_1)$, where $q_1 \geq 27$, then

$$|\mathrm{Fix}_{\mathcal{L}}(\langle i \rangle)| = \frac{b \cdot e(G_L)}{e(G)} \leq \frac{|G| \cdot e(T_L)}{|T_L| \cdot e(T)} = \frac{|C_T(i)| \cdot e(T_L) \cdot m}{|T_L|}.$$

Note that here

$$e(PSL(2, q_1)) = q_1(q_1 - 1)/2$$

and

$$e(2 \times PSL(2, q_1)) = q_1(q_1 - 1) + 1,$$

and so

$$f_2 = |\mathrm{Fix}_{\mathcal{L}}(\langle i \rangle)| < \frac{2q(q^2-1)m}{q_1 + 1}.$$

If $a = c$, then by Lemma 2.13, we have

$$v = \frac{q^3(q^3+1)(q-1)}{3^3(3^3+1)(3-1)} < \frac{4q^2(q^2-1)^2 m^2}{(q_1+1)^2} \leq \frac{4q^2(q^2-1)^2 m^2}{28^2},$$

and so

$$3^a(3^{2a} - 3^a + 1) < 8a^2(3^{2a} - 1).$$

Thus $a \neq 9$. This implies that T is line-transitive on $\mathcal{S}$.

Therefore, T must be line-transitive on $\mathcal{S}$. By [24], we have proved Theorem 1.1 given in the Introduction.

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References

[1] R.E. Block, On the orbits of collineation groups, Math. Z. 96 (1967) 33–49.

[2] F. Buekenhout, A. Delandtsheer, J. Doyen, P.B. Kleidman, M.W. Liebeck, J. Saxl, Linear spaces with flag-transitive automorphism groups, Geom. Dedicata 36 (1990) 89–94.

[3] F. Buekenhout, A. Delandtsheer, J. Doyen, Finite linear spaces with flag-transitive groups, J. Combin. Theory Ser. A 49 (1988) 268–293.

[4] A.R. Camina, Socle of automorphism-groups of linear-spaces, Bull. London. Math. Soc. 28 (1996) 269–272.

[5] A.R. Camina, P.M. Neumann, C.E. Praeger, Alternating groups acting on finite linear spaces, Proc. London Math. Soc. 87 (2003) 29–53.

[6] A.R. Camina, C.E. Praeger, Line-transitive automorphism groups of linear spaces, Bull. London Math. Soc. 25 (1993) 309–315.

[7] A.R. Camina, C.E. Praeger, Line-transitive, point quasiprimitive automorphism groups of finite linear spaces are affine or almost simple, Aequationes Math. 61 (2001) 221–232.

[8] A.R. Camina, J. Siemons, Block transitive automorphisms of $2-(v,k,1)$ block designs, J. Combin. Theory Ser. A 51 (1989) 268–276.

[9] A. Camina, F. Spiezia, Sporadic groups and automorphisms of linear spaces, J. Combin. Des. 8 (2000) 353–362.

[10] N. Gill, Line-transitive linear spaces, Ph.D. Thesis, University of Cambridge, 2005.

[11] R.M. Guralnick, P. Muller, J. Saxl, The rational function analogue of a question of Schur and exceptionality of permutation representations, Mem. Amer. Math. Soc. 162 (773) (2003).

[12] P.B. Kleidman, The Maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, the Ree groups ${}^2G_2(q)$, and their automorphism groups, J. Algebra 117 (1988) 30–71.

[13] V.M. Levchuk, Ya.N. Nuzhin, Structure of Ree groups, Algebra Logika 24 (1985) 26–41.

[14] H.L. Li, W.J. Liu, Line-primitive linear spaces with $k/(k,v) \leq 10$, J. Combin. Theory Ser. A 93 (2001) 153–167.

[15] W.J. Liu, Block transitive $2-(v,k,1)$ designs, Ph.D. Thesis, Zhejiang University, 1998.

[16] W.J. Liu, Finite linear spaces admitting a projective group $PSU(3,q)$ with q even, Linear Algebra Appl. 374 (2003) 291–305.

[17] W.J. Liu, Finite linear spaces admitting a two-dimensional projective linear group, J. Combin. Theory Ser. A 103 (2003) 209–222.

[18] W.J. Liu, Suzuki groups and automorphisms of finite linear spaces, Discrete Math. 269 (2003) 181–190.

[19] W.J. Liu, The Chevalley groups $G_2(2^n)$ and $2-(v,k,1)$ designs, Algebra Colloq. 8 (2001) 471–480.

[20] W.J. Liu, The Chevalley groups $G_2(q)$ with q odd and $2-(v,k,1)$ designs, European J. Combin. 24 (2003) 331–346.

[21] W.J. Liu, S.J. Dai, A note on Suzuki simple groups and linear spaces (in press).

[22] W.J. Liu, H.L. Li, C.G. Ma, Suzuki groups $Sz(q)$ and $2-(v,k,1)$ designs, European J. Combin. 22 (2001) 513–519.

[23] W.J. Liu, J.X. Tang, S.J. Dai, Ree groups and projective planes, Ann. of Math. (in Chinese) (in press).

[24] W.J. Liu, S.L. Zhou, H.L. Li, X.G. Fang, Finite linear spaces admitting a Ree simple group, European J. Combin. 25 (2004) 311–325.

[25] R. Ree, A family of simple groups associated with the simple Lie algebra of type (G_2), Amer. J. Math. 83 (1961) 432–462.
[26] F. Spiezia, Simple groups and automorphisms of linear spaces, Ph.D. Thesis, University of East Anglia, 1997.
[27] S. Zhou, H. Li, W. Liu, The Ree groups ${}^2G_2(q)$ and $2-(v,k,1)$ block designs, Discrete Math. 224 (2000) 251–258.